A NOTE ON MULTIPLE SUMMING OPERATORS AND APPLICATIONS

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Abstract. We prove a new result on multiple summing operators and among other results applications, we provide a new extension of Littlewood's 4/3 inequality to m-linear forms.

1. Introduction

Let $\mathbb{K}$ be the real scalar field $\mathbb{R}$ or the complex scalar field $\mathbb{C}$. As usual, for a positive integer N we define $\ell_\infty^N = \{(x_n)_{n=1}^\infty \subset \mathbb{K} \text{ bounded }\}$, $c_0 = \{(x_n)_{n=1}^\infty \subset \mathbb{K} : \lim x_n = 0\}$ and e_j represents the canonical vector of c_0 with 1 in the j-th coordinate and 0 elsewhere. Littlewood's 4/3 inequality [16], proved in 1930, asserts that

$$\left(\sum_{i,j=1}^{\infty} |U(e_i, e_j)|^{\frac{4}{3}}\right)^{\frac{3}{4}} \leq \sqrt{2}\,\|U\|$$

for every continuous bilinear form $U : c_0 \times c_0 \to \mathbb{K}$ or, equivalently,

$$\left(\sum_{i,j=1}^{N} |U(e_i, e_j)|^{\frac{4}{3}}\right)^{\frac{3}{4}} \leq \sqrt{2}\,\|U\|$$

for every positive integer N and all bilinear forms $U : \ell_\infty^N \times \ell_\infty^N \to \mathbb{K}$.

It is well known that the exponent 4/3 is optimal and it was recently shown in [13] that the constant $\sqrt{2}$ is also optimal for real scalars. For complex scalars, the constant $\sqrt{2}$ can be improved to $2/\sqrt{\pi}$, although it seems to be not known if this value is optimal. The natural step further is to investigate sums

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \left|U(e_{i_1}, \ldots, e_{i_m})\right|^{r}\right)^{\frac{1}{r}}$$

for m-linear forms $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{K}$. The exponent 4/3 need to be increased to have a similar inequality for multilinear forms; this is what the H.F. Bohnenblust and E. Hille discovered in 1931 ([8], and also [10]). More precisely, the Bohnenblust–Hille inequality asserts that for every positive integer m there is a constant $C_m \geq 1$ so that

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \left|U(e_{i_1}, \ldots, e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq C_m\,\|U\| \tag{1.1}$$

for all positive integers N and all m-linear forms $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{K}$; moreover, the exponent $2m/(m+1)$ is sharp. Another natural question is:

Is it possible to obtain multilinear versions of Littlewood's 4/3 *inequality keeping the exponent* 4/3?

This problem was treated at least in two recent papers (we state the results for complex scalars but the case of real scalars is similar, with slightly different constants):

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- ([1]) For all positive integers N and all m-linear forms $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{C}$ we have

$$\left(\sum_{i,j=1}^{N} |U(e_i, ..., e_i, e_j, ..., e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \frac{2}{\sqrt{\pi}} \|U\| .$$

- ([3]) For all positive integers N and all m-linear forms $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{C}$ we have

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \left|U(e_{i_1}, \ldots, e_{i_m})\right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \left(\prod_{j=2}^{m} \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}} \right) N^{\frac{m-2}{4}} \|U\|$$

  and the exponent $\frac{m-2}{4}$ is optimal.

In this paper we investigate this problem from a different point of view. More precisely, as a consequence of our main result we show that for all positive integers $m \geq 3$ and bijections $\sigma_1, \ldots, \sigma_{m-2}$ from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ we have

$$\left(\sum_{i,j=1}^{\infty} \left|U\left(e_i, e_j, e_{\sigma_1(i,j)}, \ldots, e_{\sigma_{m-2}(i,j)}\right)\right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2}\|U\|$$

for every continuous m-linear form $U : c_0 \times \cdots \times c_0 \to \mathbb{K}$.

We prefer to begin with the theory of multiple summing operators and state our main result in this context; then the above result (among others) will be just simple consequences of the main result.

2. Multiple summing operators

Let $E, E_1, ..., E_m$ and F denote Banach spaces over $\mathbb{K}$ and let B_{E^*} denote the closed unit ball of the topological dual of E. If $1 \leq q \leq \infty$, by q^* we represent the conjugate of q. For $p \geq 1$, by $\ell_p(E)$ we mean the space of absolutely p–summable sequences in E; also $\ell_p^w(E)$ denotes the linear space of the sequences $(x_j)_{j=1}^\infty$ in E such that $(\varphi(x_j))_{j=1}^\infty \in \ell_p$ for every continuous linear functional $\varphi : E \to \mathbb{K}$. The function

$$\left\|(x_j)_{j=1}^\infty\right\|_{w,p} = \sup_{\varphi \in B_{E^*}} \left\|(\varphi(x_j))_{j=1}^\infty\right\|_p$$

defines a norm on $\ell_p^w(E)$. The space of all continuous m-linear operators $T : E_1 \times \cdots \times E_m \to F$, with the sup norm, is denoted by $\mathcal{L}(E_1, ..., E_m; F)$.

The notion of multiple summing operators, introduced independently by Matos and Pérez-García ([17, 19]), is a natural extension of the classical notion of absolutely summing linear operators (see [14]). But multiple summing operators is certainly one of the most fruitful approaches (see [22, 23, 24] for recent papers). For different approaches we mention, for instance [9, 11, 18, 20, 21].

Definition 1. *Let $1 \leq q_1, ..., q_m \leq p < \infty$. A multilinear operator $T \in \mathcal{L}(E_1, ..., E_m; F)$ is multiple $(p; q_1, ..., q_m)$–summing if there exists a $C > 0$ such that*

$$\left(\sum_{j_1,\ldots,j_m=1}^{\infty} \left\|T\left(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}\right)\right\|^p \right)^{\frac{1}{p}} \leq C \prod_{k=1}^{m} \left\|\left(x_{j_k}^{(k)}\right)_{j_k=1}^\infty\right\|_{w,q_k}$$

for all $\left(x_j^{(k)}\right)_{j=1}^\infty \in \ell_{q_k}^w(E_k)$, $k \in \{1, ..., m\}$. We represent the class of all multiple $(p; q_1, ..., q_m)$–summing operators from $E_1, ..., E_m$ to F by $\Pi_{\mathrm{mult}(p;q_1,\ldots,q_m)}(E_1, ..., E_m; F)$ and $\pi_{\mathrm{mult}(p;q_1,\ldots,q_m)}(T)$ denotes the infimum over all C as above.

The main result of this section is the following theorem. Its proof is inspired in arguments from [5, 9].

Theorem 1. *Let $n > m \geq 1$ be positive integers and $E_1, ..., E_n, F$ Banach spaces. If*

$$\Pi_{\mathrm{mult}(p;q_1,\ldots,q_m)}(E_1, ..., E_m; F) = \mathcal{L}(E_1, ..., E_m; F), \tag{2.1}$$

then there is a constant $C > 0$ (not depending on n) such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} \left\| U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},...,x_{i_1\cdots i_m}^{(n)}) \right\|^p \right)^{\frac{1}{p}}$$
$$\leq C\|U\| \prod_{k=1}^{m} \left\| \left(x_i^{(k)} \right)_{i=1}^{\infty} \right\|_{w,q_k} \prod_{k=m+1}^{n} \left\| \left(x_{i_1\cdots i_m}^{(k)} \right)_{i_1,\ldots,i_m=1}^{\infty} \right\|_{w,1},$$

for all n-linear forms $U : E_1 \times \cdots \times E_n \to F$.

Proof. The case $m = 1$ is known (see [9, Corollary 3.3]). For $m \geq 2$ let us proceed by induction on n. First we will show that the result holds for $n = m+1$. Let N be a positive integer and $x_{i_1\cdots i_m}^{(m+1)} \in E_{m+1}$. By the Hahn–Banach theorem we can choose norm one functionals $\varphi_{i_1\cdots i_m}$ such that

$$\left\| U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)}) \right\| = \varphi_{i_1\cdots i_m} \left(U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)}) \right)$$

for all $i_1,...,i_m = 1,\ldots,N$.

A duality argument gives us non-negative real numbers $\alpha_{i_1\cdots i_m}$ such that

$$\sum_{i_1,\ldots,i_m=1}^{N} \alpha_{i_1\cdots i_m}^{p^*} = 1,$$

where p^* is the conjugate number of p, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$, and

$$\begin{aligned}
&\left(\sum_{i_1,\ldots,i_m=1}^{N} \left\| U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)}) \right\|^p \right)^{\frac{1}{p}} \\
&= \sum_{i_1,\ldots,i_m=1}^{N} \alpha_{i_1\cdots i_m} \left\| U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)}) \right\| \\
&= \sum_{i_1,\ldots,i_m=1}^{N} \alpha_{i_1\cdots i_m} \varphi_{i_1\cdots i_m} \left(U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)}) \right).
\end{aligned}$$

Let $r_{j_1\cdots j_m}$ be the Rademacher functions indexed on $\mathbb{N} \times \cdots \times \mathbb{N}$ (the order is not important). We have

$$\begin{aligned}
&\int_0^1 \sum_{i_1,\ldots,i_m=1}^{N} r_{i_1\cdots i_m}(t) \alpha_{i_1\cdots i_m} \varphi_{i_1\cdots i_m} \left(U(x_{i_1}^{(1)},...,x_{i_m}^{(m)}, \sum_{j_1,\ldots,j_m=1}^{N} r_{j_1\cdots j_m}(t) x_{j_1\cdots j_m}^{(m+1)}) \right) dt \\
&= \sum_{i_1,\ldots,i_m=1}^{N} \sum_{j_1,\ldots,j_m=1}^{N} \alpha_{i_1\cdots i_m} \varphi_{i_1\cdots i_m} \left(U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{j_1\cdots j_m}^{(m+1)}) \right) \int_0^1 r_{i_1\cdots i_m}(t) r_{j_1\cdots j_m}(t)\, dt \\
&= \sum_{i_1,\ldots,i_m=1}^{N} \alpha_{i_1\cdots i_m} \varphi_{i_1\cdots i_m} \left(U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)}) \right) \\
&= \left(\sum_{i_1,\ldots,i_m=1}^{N} \left\| U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)}) \right\|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Hence

$$
\begin{aligned}
&\left(\sum_{i_1,\dots,i_m=1}^{N} \left\| U(x_{i_1}^{(1)}, ..., x_{i_m}^{(m)}, x_{i_1\cdots i_m}^{(m+1)}) \right\|^p \right)^{\frac{1}{p}} \\
&\le \int_0^1 \left| \sum_{i_1,\dots,i_m=1}^{N} r_{i_1\cdots i_m}(t)\alpha_{i_1\cdots i_m}\varphi_{i_1\cdots i_m}\left(U(x_{i_1}^{(1)}, ..., x_{i_m}^{(m)}, \sum_{j_1,\dots,j_m=1}^{N} r_{j_1\cdots j_m}(t)x_{j_1\cdots j_m}^{(m+1)}) \right) \right| dt \\
&\le \sup_{t\in[0,1]} \left| \sum_{i_1,\dots,i_m=1}^{N} r_{i_1\cdots i_m}(t)\alpha_{i_1\cdots i_m}\varphi_{i_1\cdots i_m}\left(U(x_{i_1}^{(1)}, ..., x_{i_m}^{(m)}, \sum_{j_1,\dots,j_m=1}^{N} r_{j_1\cdots j_m}(t)x_{j_1\cdots j_m}^{(m+1)}) \right) \right| \\
&\le \sup_{t\in[0,1]} \sum_{i_1,\dots,i_m=1}^{N} \alpha_{i_1\cdots i_m} \left\| U\left(x_{i_1}^{(1)}, ..., x_{i_m}^{(m)}, \sum_{j_1,\dots,j_m=1}^{N} r_{j_1\cdots j_m}(t)x_{j_1\cdots j_m}^{(m+1)} \right) \right\| \\
&\le \left(\sum_{i_1,\dots,i_m=1}^{N} \alpha_{i_1\cdots i_m}^{p^*} \right)^{\frac{1}{p^*}} \cdot \sup_{t\in[0,1]} \left(\sum_{i_1,\dots,i_m=1}^{N} \left\| U(x_{i_1}^{(1)}, ..., x_{i_m}^{(m)}, \sum_{j_1,\dots,j_m=1}^{N} r_{j_1\cdots j_m}(t)x_{j_1\cdots j_m}^{(m+1)}) \right\|^p \right)^{\frac{1}{p}} \\
&\le \sup_{t\in[0,1]} \pi_{(p;q_1,\dots,q_m)}\left(U(\cdot, ..., \cdot, \sum_{j_1,\dots,j_m=1}^{N} r_{j_1\cdots j_m}(t)x_{j_1\cdots j_m}^{(m+1)}) \right) \prod_{k=1}^{m} \left\| \left(x_i^{(k)} \right)_{i=1}^{N} \right\|_{w,q_k}
\end{aligned}
$$

where in the last inequality we have used (2.1). From (2.1) it follows from the Open Mapping Theorem that there is a constant $C > 0$ such that $\pi_{(p;q_1,\dots,q_m)}(\,\cdot\,) \le C\|\cdot\|$. Then

$$
\begin{aligned}
&\left(\sum_{i_1,\dots,i_m=1}^{N} \left\| U(x_{i_1}^{(1)}, ..., x_{i_m}^{(m)}, x_{i_1\cdots i_m}^{(m+1)}) \right\|^p \right)^{\frac{1}{p}} \\
&\le \sup_{t\in[0,1]} C \left\| U(\cdot, ..., \cdot, \sum_{j_1,\dots,j_m=1}^{N} r_{j_1\cdots j_m}(t)x_{j_1\cdots j_m}^{(m+1)}) \right\| \prod_{k=1}^{m} \left\| \left(x_i^{(k)} \right)_{i=1}^{N} \right\|_{w,q_k} \\
&\le C\|U\| \sup_{t\in[0,1]} \left\| \sum_{i_1,\dots,i_m=1}^{N} r_{i_1\cdots i_m}(t)x_{i_1\cdots i_m}^{(m+1)} \right\| \prod_{k=1}^{m} \left\| \left(x_i^{(k)} \right)_{i=1}^{N} \right\|_{w,q_k} \\
&\le C\|U\| \left(\prod_{k=1}^{m} \left\| \left(x_i^{(k)} \right)_{i=1}^{N} \right\|_{w,q_k} \right) \left\| \left(x_{i_1\cdots i_m}^{(m+1)} \right)_{i_1,\dots,i_m=1}^{N} \right\|_{w,1}.
\end{aligned}
$$

The proof is completed by an induction argument, as follows. Suppose that the result is valid for a positive integer $n \ge m+1$. Let N be a positive integer and E_{n+1} a Banach space. Let $x_{i_1\cdots i_m}^{(n+1)} \in E_{n+1}$ and norm one functionals $\varphi_{i_1\cdots i_m}$ such that

$$
\begin{aligned}
&\left\| U(x_{i_1}^{(1)}, ..., x_{i_m}^{(m)}, x_{i_1\cdots i_m}^{(m+1)}, \dots, x_{i_1\cdots i_m}^{(n+1)}) \right\| \\
&= \varphi_{i_1\cdots i_m}\left(U(x_{i_1}^{(1)}, ..., x_{i_m}^{(m)}, x_{i_1\cdots i_m}^{(m+1)}, \dots, x_{i_1\cdots i_m}^{(n+1)}) \right),
\end{aligned}
$$

for all $i_1, ..., i_m = 1, \dots, N$. A duality argument gives us non-negative real numbers $\alpha_{i_1\cdots i_m}$ such that

$$
\sum_{i_1,\dots,i_m=1}^{N} \alpha_{i_1\cdots i_m}^{p^*} = 1
$$

and

$$\begin{aligned}
&\left(\sum_{i_1,\dots,i_m=1}^{N}\left\|U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},\dots,x_{i_1\cdots i_m}^{(n+1)})\right\|^p\right)^{\frac{1}{p}}\\
&=\sum_{i_1,\dots,i_m=1}^{N}\alpha_{i_1\cdots i_m}\left\|U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},\dots,x_{i_1\cdots i_m}^{(n+1)})\right\|\\
&=\sum_{i_1,\dots,i_m=1}^{N}\alpha_{i_1\cdots i_m}\varphi_{i_1\cdots i_m}\left(U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},\dots,x_{i_1\cdots i_m}^{(n+1)})\right).
\end{aligned}$$

We also have

$$\begin{aligned}
&\int_0^1\sum_{i_1,\dots,i_m=1}^{N}r_{i_1\cdots i_m}(t)\alpha_{i_1\cdots i_m}\\
&\quad\times\varphi_{i_1\cdots i_m}\left(U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},\dots,x_{i_1\cdots i_m}^{(n)},\sum_{j_1,\dots,j_m=1}^{N}r_{j_1\cdots j_m}(t)x_{j_1\cdots j_m}^{(n+1)})\right)dt\\
&=\sum_{i_1,\dots,i_m=1}^{N}\sum_{j_1,\dots,j_m=1}^{N}\alpha_{i_1\cdots i_m}\varphi_{i_1\cdots i_m}\left(U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},\dots,x_{i_1\cdots i_m}^{(n)},x_{j_1\cdots j_m}^{(n+1)})\right)\\
&\quad\times\int_0^1 r_{i_1\cdots i_m}(t)r_{j_1\cdots j_m}(t)\,dt\\
&=\sum_{i_1,\dots,i_m=1}^{N}\alpha_{i_1\cdots i_m}\varphi_{i_1\cdots i_m}\left(U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},\dots,x_{i_1\cdots i_m}^{(n+1)})\right)\\
&=\left(\sum_{i_1,\dots,i_m=1}^{N}\left\|U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},\dots,x_{i_1\cdots i_m}^{(n+1)})\right\|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence using the induction hypothesis

$$
\begin{aligned}
&\left(\sum_{i_1,\dots,i_m=1}^{N}\left\|U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},\dots,x_{i_1\cdots i_m}^{(n+1)})\right\|^{p}\right)^{\frac{1}{p}} \\
&\leq \int_0^1 \Bigg| \sum_{i_1,\dots,i_m=1}^{N} r_{i_1\cdots i_m}(t)\alpha_{i_1\cdots i_m} \\
&\quad \times\varphi_{i_1\cdots i_m}\left(U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},\dots,x_{i_1\cdots i_m}^{(n)},\sum_{j_1,\dots,j_m=1}^{N} r_{j_1\cdots j_m}(t)x_{j_1\cdots j_m}^{(n+1)})\right)\Bigg| \, dt \\
&\leq \sup_{t\in[0,1]}\sum_{i_1,\dots,i_m=1}^{N}\alpha_{i_1\cdots i_m}\left\|U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},\dots,x_{i_1\cdots i_m}^{(n)},\sum_{j_1,\dots,j_m=1}^{N} r_{j_1\cdots j_m}(t)x_{j_1\cdots j_m}^{(n+1)})\right\| \\
&\leq \left(\sum_{i_1,\dots,i_m=1}^{N}\alpha_{i_1\cdots i_m}^{p^*}\right)^{\frac{1}{p^*}}\cdot\sup_{t\in[0,1]}\Bigg(\sum_{i_1,\dots,i_m=1}^{N}\Big\|U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},\dots,x_{i_1\cdots i_m}^{(n)}, \\
&\qquad \sum_{j_1,\dots,j_m=1}^{N} r_{j_1\cdots j_m}(t)x_{j_1\cdots j_m}^{(n+1)})\Big\|^{p}\Bigg)^{\frac{1}{p}} \\
&\leq \sup_{t\in[0,1]} C\left\|U\left(\cdot,...,\cdot,\sum_{j_1,\dots,j_m=1}^{N} r_{j_1\cdots j_m}(t)x_{j_1\cdots j_m}^{(n+1)}\right)\right\| \\
&\qquad \times\left(\prod_{k=1}^{m}\left\|\left(x_i^{(k)}\right)_{i=1}^{N}\right\|_{w,q_k}\right)\left(\prod_{k=m+1}^{n}\left\|\left(x_{i_1\cdots i_m}^{(k)}\right)_{i_1,\dots,i_m=1}^{N}\right\|_{w,1}\right) \\
&\leq C\left\|U\right\|\left(\prod_{k=1}^{m}\left\|\left(x_i^{(k)}\right)_{i=1}^{N}\right\|_{w,q_k}\right)\left(\prod_{k=m+1}^{n+1}\left\|\left(x_{i_1\cdots i_m}^{(k)}\right)_{i_1,\dots,i_m=1}^{N}\right\|_{w,1}\right).
\end{aligned}
$$

Now we just make $N\to\infty$. $\square$

Example 1. *If F is a Banach space with cotype 2 it is well known that $\Pi_{\mathrm{mult}(2;2,\dots,2)}\left({}^{m}c_0;F\right)=\mathcal{L}\left({}^{m}c_0;F\right)$. From the above theorem we conclude that*

$$
\begin{aligned}
&\left(\sum_{i_1,\dots,i_m=1}^{\infty}\left\|U(x_{i_1}^{(1)},...,x_{i_m}^{(m)},x_{i_1\cdots i_m}^{(m+1)},...,x_{i_1\cdots i_m}^{(n)})\right\|^{2}\right)^{\frac{1}{2}} \\
&\qquad\leq C\|U\|\prod_{k=1}^{m}\left\|\left(x_i^{(k)}\right)_{i=1}^{\infty}\right\|_{w,2}\prod_{k=m+1}^{n}\left\|\left(x_{i_1\cdots i_m}^{(k)}\right)_{i_1,\dots,i_m=1}^{\infty}\right\|_{w,1},
\end{aligned}
$$

regardless of the Banach space E and regardless of the n-linear operator $U: c_0\times\overset{m\ times}{\cdots}\times c_0\times E\times\cdots\times E\to F$.

Remark 1. *The constant C that appears in the above theorem can be chosen as the constant from the Open Mapping Theorem used in the coincidence (2.1).*

Remark 2. *The case $F=\mathbb{K}$ and $m=1$ with $q=p=1$ recovers the Defant-Voigt Theorem (see* [2]*).*

Remark 3. *Theorem 1 is in some sense optimal. In fact it was recently proved in* [4] *that the Defant-Voigt Theorem is optimal in the following sense: every continuous m-linear form is absolutely $(1;1,...,1)$-summing and this result can not be improved to $(p;1,...,1)$-summing with $p<1$.*

3. Some applications

In this section we show how the result proved in the previous section is connected to the problem stated in the introduction of this note.

3.1. **Variations of Littlewood's 4/3 theorem and Bohnenblust–Hille inequality.** We begin by proving the result stated in the Introduction:

Theorem 2. *Let $m \geq 3$ be an integer and $\sigma_1, \dots, \sigma_{m-2}$ be bijections from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. Then*

$$\left(\sum_{i,j=1}^{\infty} \left| U\left(e_i, e_j, e_{\sigma_1(i,j)}, \dots, e_{\sigma_{n-2}(i,j)}\right) \right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2} \|U\| \tag{3.1}$$

for all continuous m-linear forms $U : c_0 \times \cdots \times c_0 \to \mathbb{K}$.

Proof. From Littlewood's 4/3 theorem we know that $\Pi_{\mathrm{mult}(4/3;1,1)}\left({}^2c_0;\mathbb{K}\right) = \mathcal{L}\left({}^2c_0;\mathbb{K}\right)$ and the constant involved is $\sqrt{2}$ (or $2/\sqrt{\pi}$ for complex scalars). By choosing $x_{ij}^{(k)} = e_{\sigma_k(i,j)}$, since $\left\| \left(e_{\sigma_k(i,j)}\right)_{ij=1}^{\infty} \right\|_{w,1} = 1$ the proof is done. □

The same argument of the previous theorem can be used to prove the following more general result:

Theorem 3. *Let $n > m \geq 1$ be positive integers and $\sigma_k : \mathbb{N}^m \to \mathbb{N}$ be bijections for all $k = 1, ..., n-m$. Then there is a constant $L_m^{\mathbb{K}} \geq 1$ such that*

$$\left(\sum_{i_1,\dots,i_m=1}^{\infty} \left| U(e_{i_1}, ..., e_{i_m}, e_{\sigma_1(i_1,\dots,i_m)}, \dots, e_{\sigma_{n-m}(i_1,\dots,i_m)}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq L_m^{\mathbb{K}} \|U\|$$

for every bounded n-linear form $U : c_0 \times \cdots \times c_0 \to \mathbb{K}$.

Remark 4. *As a matter of fact, the constants $L_m^{\mathbb{K}}$ can be estimated. From the proof of Theorem 1 it is simple to see that $L_m^{\mathbb{K}}$ can be chosen as the best known constants of the Bohnenblust–Hille inequality. So, using the estimates of* [6]*, we know that*

$$\begin{aligned} L_m^{\mathbb{C}} &\leq \prod_{j=2}^{m} \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}, \\ L_m^{\mathbb{R}} &\leq 2^{\frac{446381}{55440} - \frac{k}{2}} \prod_{j=14}^{k} \left(\frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}} \right)^{\frac{j}{2-2j}}, \quad \textit{for } m \geq 14, \\ L_m^{\mathbb{R}} &\leq \left(\sqrt{2}\right)^{\sum_{j=1}^{k-1} \frac{1}{j}}, \quad \textit{for } 2 \leq m \leq 13. \end{aligned} \tag{3.2}$$

The above estimates can be rewritten as (see [6])

$$\begin{aligned} L_m^{\mathbb{C}} &< m^{0.21139}, \\ L_m^{\mathbb{R}} &< 1.3 \times m^{0.36482}, \end{aligned}$$

The extension of the Bohnenblust–Hille inequality to ℓ_p spaces in the place of ℓ_∞ spaces is divided in two cases: $m < p \leq 2m$ and $p \geq 2m$. The case $p \geq 2m$, sometimes called Hardy–Littlewood/Praciano-Pereira inequality (see [15, 25]) states that there exists a (optimal) constant $C_{m,p}^{\mathbb{K}} \geq 1$ such that, for all positive integers N and all m-linear forms $T : \ell_p^N \times \cdots \times \ell_p^N \to \mathbb{K}$,

$$\left(\sum_{i_1,\dots,i_m=1}^{N} |T(e_{i_1}, \dots, e_{i_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{m,p}^{\mathbb{K}} \|T\|. \tag{3.3}$$

4. Final remark

When $m < p \leq 2m$ the Hardy–Littlewood inequality is also known as Hardy–Littlewood/Dimant-Sevilla–Peris inequality ([12, 15]). It reads as follows:

Theorem 4 (Hardy–Littlewood/Dimant–Sevilla-Peris). *For $m < p \leq 2m$, there is a constant $C_{\mathbb{K},m,p} \geq 1$ such that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} |T(e_{i_1},\ldots,e_{i_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq C_{\mathbb{K},m,p} \|T\|$$

for all positive integers N and all m-linear form $T : \ell_p^N \times \cdots \times \ell_p^N \to \mathbb{K}$. Moreover the exponent $\frac{p}{p-m}$ is optimal.

In this case we can prove the following (optimal) result, which does not depend on the results developed in the previous sections:

Proposition 1. *Let $m > n \geq 1$ be positive integers, let $m < p \leq 2m$ and $\sigma_k : \mathbb{N}^n \to \mathbb{N}$ be bijections for all $k = 1, ..., n$. Then there is a constant $C_{\mathbb{K},m,n,p} \geq 1$ such that*

$$\left(\sum_{i_1,\ldots,i_n=1}^{N} \left| U(e_{i_1}, ..., e_{i_n}, e_{\sigma_1(i_1,\ldots,i_n)}, \ldots, e_{\sigma_{m-n}(i_1,\ldots,i_n)}) \right|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq C_{\mathbb{K},m,n,p} \|U\|$$

for all positive integers $N \geq 1$ and all continuous m-linear form $U : \ell_p^N \times \cdots \times \ell_p^N \to \mathbb{K}$. Moreover, the exponent $p/(p-m)$ is optimal.

Proof. For the sake of simplicity we suppose $n = 2$. The general case is similar. Note that, using Theorem 4 we have

$$\begin{aligned} &\left(\sum_{i,j=1}^{N} \left| U(e_i, e_j, e_{\sigma_1(i,j)}, \ldots, e_{\sigma_{m-2}(i,j)}) \right|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \\ &\leq \left(\sum_{i_1,\ldots,i_m=1}^{\infty} |U(e_{i_1},\ldots,e_{i_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq C_{\mathbb{K},m,p} \|U\| . \end{aligned}$$

The optimality of the exponent $\frac{p}{p-m}$ is proved next using the same argument of the proof of the theorem of Hardy–Littlewood/Dimant-Sevilla-Peris (see [12, 15]). Consider the m-linear form

$$U : \ell_p \times \cdots \times \ell_p \to \mathbb{K}$$

given by

$$U(x^{(1)}, ..., x^{(m)}) = \sum_{i=1}^{N} x_i^{(1)} x_i^{(2)} x_{\sigma_1(i,i)}^{(3)} \cdots x_{\sigma_{m-2}(i,i)}^{(m)}.$$

From Hölder's inequality we have

$$\|U\| \leq N^{\frac{p-m}{p}}.$$

If the theorem is valid for a power s, then

$$\begin{aligned} &\left(\sum_{i=1}^{N} \left| U\left(e_i, e_i, e_{\sigma_1(i,i)}, \ldots, e_{\sigma_{m-2}(i,i)}\right) \right|^{s} \right)^{\frac{1}{s}} \\ &= \left(\sum_{i,j=1}^{N} \left| U\left(e_i, e_j, e_{\sigma_1(i,j)}, \ldots, e_{\sigma_{m-2}(i,j)}\right) \right|^{s} \right)^{\frac{1}{s}} \\ &\leq C_{\mathbb{K},m,p} \|U\| \leq C_{\mathbb{K},m,p} N^{\frac{p-m}{p}} \end{aligned}$$

and thus

$$N^{\frac{1}{s}} \leq C_{\mathbb{K},m,p} N^{\frac{p-m}{p}}$$

and hence

$$s \geq \frac{p}{p-m}.$$

□

It is important to recall that a somewhat similar inequality due to Zalduendo asserts that

$$\left(\sum_{i=1}^{n}|T(e_i,\dots,e_i)|^{\frac{p}{p-m}}\right)^{\frac{p-m}{p}}\leq\|T\|$$

for all positive integers n and all m-linear forms $T:\ell_p^n\times\cdots\times\ell_p^n\to\mathbb{K}$ and the exponent $\frac{p}{p-m}$ is optimal. Note that Zalduendo's result and Proposition 1 are slightly different.

5. Appendix: a variation of the Kahane–Salem–Zygmund and applications

In this section we follow a method of [7] to prove this. Let us denote by $\psi_2(x):=\exp(x^2)-1$ for $x\geq 0$. Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability measure space and let us consider the Orlicz space $L_{\psi_2}=L_{\psi_2}(\Omega,\mathcal{A},\mathbb{P})$ associated to ψ_2 formed by all real-valued random variables X on $(\Omega,\mathcal{A},\mathbb{P})$ such that $\mathbb{E}\left(\psi_2\left(|X|/c\right)\right)<\infty$ for some $c>0$. The associated Orlicz norm $\|\cdot\|_{\psi_2}$ is given by

$$\|X\|_{\psi_2}:=\inf\{c>0\,;\,\mathbb{E}\left(\psi_2\left(|X|/c\right)\right)\leq 1\},$$

and $(L_{\psi_2},\|\cdot\|_{\psi_2})$ is a Banach space. We shall use the following lemma, which was suggested to us by F. Bayart.

Lemma 1. *Let M be a metric space and let $(X(\omega,x))$ a family of random variables defined on $(\Omega,\mathcal{A},\mathbb{P})$ and indexed by M. Assume that there exists $A>0$ and a finite set $F\subset M$ such that*

i. *For any $x\in M$, $\|X(\cdot,x)\|_{\psi_2}\leq A$;*
ii. *For any $x\in M$, there exists $y\in F$ such that*

$$\sup_{\omega\in\Omega}|X(\omega,x)-X(\omega,y)|\leq\frac{1}{2}\sup_{z\in M}|X(\omega,z)|.$$

Then for any $R>0$ with $\frac{card(F)}{\psi_2(R/A)}<1$, there exists $\omega\in\Omega$ satisfying

$$\sup_{x\in M}|X(\omega,x)|\leq 2R.$$

Proof. This is exactly what is done in [7], Step 2 and Step 3 of the proof of Theorem 3.1, in an abstract context. For the sake of completeness, we give the details. Given $x\in M$, condition (ii) provides us $y\in F$ such that

$$\sup_{\omega\in\Omega}|X(\omega,x)-X(\omega,y)|\leq\frac{1}{2}\sup_{z\in M}|X(\omega,z)|.$$

From

$$|X(\omega,x)|\leq|X(\omega,x)-X(\omega,y)|+|X(\omega,y)|\leq\frac{1}{2}\sup_{z\in M}|X(\omega,z)|+\sup_{w\in F}|X(\omega,w)|$$

we get that, for any $\omega\in\Omega$,

$$\sup_{x\in M}|X(\omega,x)|\leq 2\sup_{w\in F}|X(\omega,w)|. \tag{5.1}$$

Let us fix $R>0$. As in the Step (3) of [7, Theorem 3.1] we have

$$\mathbb{P}\left(\{\omega\in\Omega\,;\,|X(\omega,x)|>R\}\right)=\mathbb{P}\left(\left\{\omega\in\Omega\,;\,\psi_2\left(\frac{|X(\omega,x)|}{A}\right)>\psi_2\left(\frac{R}{A}\right)\right\}\right)$$

The Markov inequality leads us to

$$\mathbb{P}\left(\{\omega\in\Omega\,;\,|X(\omega,x)|>R\}\right)\leq\frac{\mathbb{E}\left(\psi_2\left(|X(\omega,x)|/A\right)\right)}{\psi_2(R/A)}.$$

Condition (i) provides $\|X(\cdot,x)\|_{\psi_2}\leq A$, thus the definition of $\|\cdot\|_{\psi_2}$ assures that $\mathbb{E}\left(\psi_2\left(|X(\omega,x)|/A\right)\right)\leq 1$. Consequently, we get that for any $\omega\in\Omega$,

$$\mathbb{P}\left(\{\omega\in\Omega\,;\,|X(\omega,x)|>R\}\right)\leq\frac{1}{\psi_2(R/A)}.$$

Since $F\subset M$ is finite,

$$\mathbb{P}\left(\left\{\omega\in\Omega\,;\,\sup_{w\in F}|X(\omega,w)|>R\right\}\right)\leq\frac{card\,F}{\psi_2(R/A)}.$$

Combining this with (5.1) we get

$$\mathbb{P}\left(\left\{\omega \in \Omega\,;\ \sup_{x\in M} |X(\omega,x)| > 2R\right\}\right) \leq \frac{card\, F}{\psi_2(R/A)}.$$

Thus, if we take $R > 0$ such that $\dfrac{\text{card}(F)}{\psi_2(R/A)} < 1$, then

$$\mathbb{P}\left(\left\{\omega \in \Omega\,;\ \sup_{x\in M} |X(\omega,x)| \leq 2R\right\}\right) > 0.$$

Therefore, there exists $\omega \in \Omega$ satisfying

$$\sup_{x\in M} |X(\omega,x)| \leq 2R.$$

□

The previous approach can be applied in the following situation: let $N \geq 1$ and let $(\varepsilon_i)_{i\in\{1,\ldots,N\}^k}$ be a sequence of independent Bernoulli variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let M be the unit ball of $(\ell_\infty^N)^n$ (endowed with the sup norm). For $x = (x^{(1)}, \ldots, x^{(n)})$ in M we define for positive integers $n_1 + \cdots + n_k = n$ and $j_l = n_1 + \cdots + n_l$, $l = 1, \ldots, k$

$$X(\omega,x) = \sum_{i\in\{1,\ldots,N\}^k} \varepsilon_i(\omega) x_{i_1}^{(1)} \cdots x_{i_1}^{(j_1)} x_{i_2}^{(j_1+1)} \cdots x_{i_2}^{(j_2)} \cdots x_{i_k}^{(j_{k-1}+1)} \cdots x_{i_k}^{(j_k)}$$

For a fixed value of x, the L^2-norm of this random process can be majorized, using the Khinchin inequality:

$$\begin{aligned}
\|X(\cdot,x)\|_2 &= \left(\int_\Omega |X(w,x)|^2\, d\mathbb{P}\right)^{1/2} \\
&= \left(\int_\Omega \left|\sum_{i\in\{1,\ldots,N\}^k} \varepsilon_i(\omega) x_{i_1}^{(1)} \cdots x_{i_1}^{(j_1)} x_{i_2}^{(j_1+1)} \cdots x_{i_2}^{(j_2)} \cdots x_{i_k}^{(j_{k-1}+1)} \cdots x_{i_k}^{(j_k)}\right|^2 d\mathbb{P}\right)^{1/2} \\
&\leq \left(\sum_{i\in\{1,\ldots,N\}^k} \left|x_{i_1}^{(1)} \cdots x_{i_1}^{(j_1)} x_{i_2}^{(j_1+1)} \cdots x_{i_2}^{(j_2)} \cdots x_{i_k}^{(j_{k-1}+1)} \cdots x_{i_k}^{(j_k)}\right|^2\right)^{1/2} \\
&\leq \left\| \left(x_{i_1}^{(j_1)}\right)_{i_1=1}^N \right\|_2 \cdots \left\| \left(x_{i_k}^{(j_k)}\right)_{i_k=1}^N \right\|_2 \\
&\leq N^{k/2}.
\end{aligned}$$

Since the ψ_2-norm of a Rademacher process is dominated by its L^2-norm, we get

$$\|X(\cdot,x)\|_{\psi_2} \leq CN^{k/2} := A$$

for some absolute constant $C > 0$.

Now, let $\delta > 0$. For a fixed value of ω, for any x, y in M with $\|x - y\| < \delta$, the multilinearity of $X(\omega,\cdot)$ ensures that

$$|X(\omega,x) - X(\omega,y)| \leq n\delta \sup_{z\in M} |X(\omega,z)|.$$

We set again $\delta = \frac{1}{2n}$ and so

$$|X(\omega,x) - X(\omega,y)| \leq \frac{1}{2} \sup_{z\in M} |X(\omega,z)|.$$

Repeating the previous argument and we observe that there exists a δ-net F of M with cardinal less than $\left(1 + \frac{2}{\delta}\right)^{2nN} = (1 + 4n)^{2nN}$ (the product nN is the dimension of $(\ell_\infty^N)^n$). Setting $R = \lambda N^{(k+1)/2}$ for some large λ (not depending on N but eventually depending on n), we obtain

$$\frac{\text{card}(F)}{\psi_2(R/A)} = \frac{(1+4n)^{2nN}}{e^{\left(\frac{\lambda N^{(k+1)/2}}{CN^{k/2}}\right)^2} - 1} = \frac{(1+4n)^{2nN}}{e^{\left(\frac{\lambda^2 N}{C^2}\right)} - 1} < 1$$

and from the lemma there exists $\omega_0 \in \Omega$ such that, for any $x \in M$,

$$|X(\omega_0, x)| \leq 2R = 2\lambda N^{\frac{k+1}{2}},$$

i.e.,

$$\|X(\omega_0, \cdot)\| \leq 2\lambda N^{\frac{k+1}{2}}.$$

Now consider an n-linear operator $U : c_0 \times \cdots \times c_0 \to \mathbb{K}$ and if

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \left| U\left(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k}\right)\right|^r \right)^{\frac{1}{r}} \leq C \|U\|$$

for all U and all positive integers N, where $\left(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k}\right)$ means $(e_{i_1}, \overset{n_1 \text{ times}}{\ldots}, e_{i_1}, \ldots, e_{i_k}, \overset{n_k \text{ times}}{\ldots}, e_{i_k})$, we have

$$\begin{aligned} N^{\frac{k}{r}} &= \left(\sum_{i \in \{1,\ldots,N\}^k} \left| \varepsilon_i(\omega) x_{i_1}^{(1)} \cdots x_{i_1}^{(j_1)} x_{i_2}^{(j_1+1)} \cdots x_{i_2}^{(j_2)} \cdots x_{i_k}^{(j_{k-1}+1)} \cdots x_{i_k}^{(j_k)} \right|^r \right)^{\frac{1}{r}} \\ &\leq C \|X(\omega_0, x)\| \\ &\leq 2C\lambda N^{\frac{k+1}{2}}. \end{aligned}$$

Making $N \to \infty$ we conclude that $r \geq 2k/(k+1)$. This result provides the optimality, for instance, of [1, Corollary 2.5]. We also recall that the case $k = n$ recovers the classical Kahane–Salem–Zygmund–Inequality.

References

[1] N. Albuquerque, G. Araújo, D. Nunez-Alarcon, D. Pellegrino, P. Rueda, Summability of multilinear operators: a unified theory and consequences, arXiv:1409.6769 [math.FA] 5 Dec 2014.

[2] R. Alencar and M.C. Matos, Some Classes of Multilinear Mappings between Banach Spaces, Publicaciones del Departamento de Análisis Matemático de la U.C.M., Sec. 1, No. 12. Madrid, 1989.

[3] G. Araújo and D. Pellegrino, Optimal Hardy–Littlewood type inequalities for m-linear forms on ℓ_p spaces with $1 \leq p \leq m$, arXiv:1502.01522 [math.FA] 11 Feb 2015.

[4] G. Araújo and D. Pellegrino, Spaceability and optimal estimates for summing multilinear operators, arXiv:1403.6064v3 [math.FA] 14 Feb 2015.

[5] R. Aron, M. Lacruz, R. Ryan and A. Tonge, The generalized Rademacher functions, Note di Mat. (1992), 15–25.

[6] F. Bayart, D. Pellegrino, and J.B. Seoane-Sepúlveda, The Bohr radius of the n–dimensional polydisc is equivalent to $\sqrt{(\log n)/n}$, Advances in Math. **264** (2014) 726–746.

[7] F. Bayart, Maximum modulus of random polynomials, Q. J. Math. **63** (2012), no. 1, 21–39.

[8] H.F. Bohnenblust and E. Hille, On the absolute convergence of Dirichlet series, Ann. of Math. (2) **32** (1931), 600–622.

[9] G. Botelho and D. Pellegrino, Coincidence situations for absolutely summing non-linear mappings, Port. Math. **64** (2007), 175–191.

[10] A. Defant, P. Sevilla-Peris, The Bohnenblust-Hille cycle of ideas from a modern point of view. Funct. Approx. Comment. Math. **50** (2014), no. 1, 55–127.

[11] V. Dimant, Strongly p-summing multilinear operators. J. Math. Anal. Appl. 278 (2003), no. 1, 182–193.

[12] V. Dimant and P. Sevilla–Peris, Summation of coefficients of polynomials on ℓ_p spaces, arXiv:1309.6063v1 [math.FA].

[13] D. Diniz, G.A. Muñoz-Fernández, D. Pellegrino and J.B. Seoane-Sepúlveda, Lower bounds for the constants in the Bohnenblust–Hille inequality: the case of real scalars, Proc. Amer. Math. Soc., accepted for publication.

[14] J. Diestel, H. Jarchow and A. Tonge, Absolutely summing operators, Cambridge University Press, Cambridge, 1995.

[15] G. Hardy and J. E. Littlewood, Bilinear forms bounded in space $[p, q]$, Quart. J. Math. **5** (1934), 241–254.

[16] J.E. Littlewood, On bounded bilinear forms in an infinite number of variables, Q. J. Math. Oxford **1** (1930), 164–174.

[17] M. C. Matos, Fully absolutely summing mappings and Hilbert Schmidt operators, Collect. Math. **54** (2003) 111–136.

[18] D. Pérez–García, Comparing different classes of absolutely summing multilinear operators, Arch. Math. (Basel) **85** (2005), 258–267.

[19] D. Pérez-García, Operadores multilineales multipl summantes, PhD Thesis, Universidad Complutense de Madrid, 2003.

[20] D. Pellegrino, P. Rueda, E. A. SÃ¡nchez PÃ©rez, Surveying the spirit of absolute summability on multilinear operators and homogeneous polynomials. Submitted.

[21] P. Rueda, E. A. Sanchez Perez, Factorization of p-dominated polynomials through Lp-spaces. Michigan Math. J. **63** (2014), no. 2, 345–353.

[22] D. Popa, A new distinguishing feature for summing, versus dominated and multiple summing operators. Arch. Math. (Basel) 96 (2011), no. 5, 455–462.

[23] D. Popa, Multiple summing, dominated and summing operators on a product of l_1 spaces. Positivity 18 (2014), no. 4, 751–765.

[24] D. Popa, Remarks on multiple summing operators on $C(\Omega)$-spaces. Positivity 18 (2014), no. 1, 29–39.
[25] T. Praciano–Pereira, On bounded multilinear forms on a class of ℓ_p spaces. J. Math. Anal. Appl. **81** (1981), 561–568.

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